Complex numbers. $a+bi$. a, b real numbers

① Multiplication: $(a+bi)\cdot(c+di)$

$$= ac + bci + adi + bdi^2$$

i satisfies $i^2=-1$

$$= ac - bd + (ad+bc)i$$

Examples: $(1+i)(2-i)$

$$= 2+1+(2i-1i) = 3+i$$

$$i(3+i)$$

$$= 3i - 1 = -1+3i$$

$$(\cos\theta + i\sin\theta)(1-2i)$$

$$= \cos\theta + 2\sin\theta + i(\sin\theta - 2\cos\theta)$$

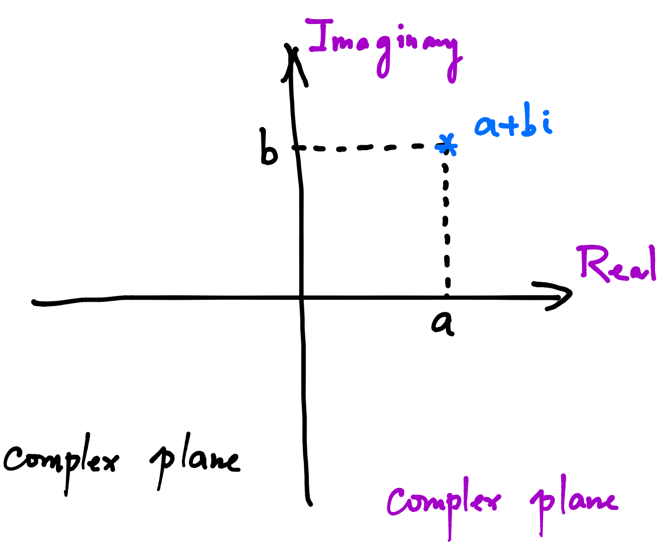
② Geometric Interpretation.

$a+bi \leftrightarrow (a, b)$

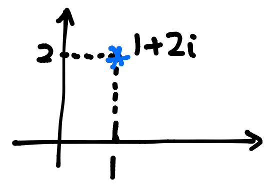
Real part $\leftrightarrow$ x-coordinate

Imaginary part $\leftrightarrow$ y-coordinate

Complex number $\leftrightarrow$ Point in the complex plane



Example: $1+2i$



③ Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

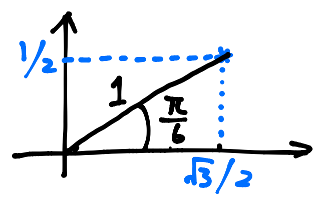
Moreover, $re^{i\theta} = r\cos\theta + i\,r\sin\theta$. $r \geq 0$

Indeed, this formula is nothing but the polar coordinate transformation

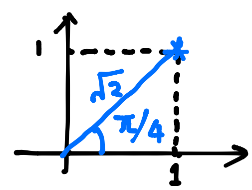
$$(r, \theta) \longleftrightarrow (r\cos\theta, r\sin\theta)$$

$$re^{i\theta} \longleftrightarrow r\cos\theta + i\,r\sin\theta.$$

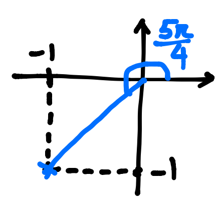
Examples: $e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.



$1+i = \sqrt{2}\,e^{i\frac{\pi}{4}}$



$-1-i = \sqrt{2}\,e^{\frac{5\pi}{4}i}$



④ Taking roots.

Notice: $e^{i\theta} = e^{i(\theta+2k\pi)}$ b/c $\cos(\theta+2k\pi) = \cos\theta$,
$\sin(\theta+2k\pi) = \sin\theta$

Example: $(1)^{\frac{1}{3}}$ means all possible complex numbers z s.t. $z^3 = 1$.

$$(1)^{\frac{1}{3}} = \pm\sqrt{1} = \pm 1$$

$$1 = e^{i(0+2k\pi)} \qquad \text{RHS} = \cos 2k\pi + i\sin 2k\pi = 1 + 0i = 1$$

$$(1)^{\frac{1}{3}} = \left(e^{i2k\pi}\right)^{\frac{1}{3}} = e^{i\frac{2}{3}k\pi}$$

$$= \begin{cases} e^{i\frac{2}{3}\cdot 0\pi} = 1 & k = 0, 3, 6, \cdots \\ e^{i\frac{2}{3}\cdot 1\pi} = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi = -\frac{1}{2} + \frac{\sqrt{3}}{2}i & k = 1, 4, 7, \cdots \\ e^{i\frac{2}{3}\cdot 2\pi} = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi = -\frac{1}{2} - \frac{\sqrt{3}}{2}i & k = 2, 5, 8, \cdots \end{cases}$$

In other words, 1 has three cubic roots over the complex numbers.

Attendance Quiz: $1^{\frac{1}{4}}$. $2^{\frac{1}{3}}$

Second Order linear homog. ODE w/ constant coefficients.

Recall: $ay'' + by' + cy = 0$ a, b, c real numbers

Characteristic Equation: $ar^2 + br + c = 0$

Characteristic Roots: r_1, r_2

Case I: $r_1 \neq r_2$ real, then the general solution is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Case II: $r_1 \neq r_2$ complex. Write $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

then the general solution is

$$y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t.$$

How comes the formula.

$r_1 = \alpha + i\beta$ is a characteristic root means the following complex function $\tilde{y} = e^{(\alpha+i\beta)t}$ is a solution.

In general complex function plays an important role in mathematics. But in the current scenario, we want real solutions.

Fact: If $y(t) = u(t) + i\,v(t)$ is a complex solution to

$$y'' + p(t)y' + q(t)y = 0, \quad p(t),\ q(t) \text{ real functions,}$$

then $u(t)$, $v(t)$ are solutions as well.

Pf: $y = u + iv$, $(u+iv)'' + p(u+iv)' + q(u+iv) = 0$

$$u'' + iv'' + pu' + ipv' + qu + iqv = 0$$

$$u'' + pu' + qu + i(v'' + pv' + qv) = 0 = 0 + 0i$$

Recall: $a + bi = c + di \Leftrightarrow a = c,\ b = d$

This tells $u'' + pu' + qu = 0$, $v'' + pv' + qv = 0$. $\Rightarrow u, v$ are solutions.

Write $\tilde{y}$ into the sum of real and $i\cdot$imaginary.

$$\tilde{y} = e^{(\alpha+i\beta)t} = e^{\alpha t}\cdot e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

$$= e^{\alpha t}\cos\beta t + i\,e^{\alpha t}\sin\beta t$$

From the fact above, $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$ are solutions. Easy to see, $W(e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t) \not\equiv 0$. By the principle of superposition, the general solution is $y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t$.

Examples: 1) $y'' + y = 0$

char. eqn: $r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$

This is how i was defined historically.

General sol'n: $y = C_1 \cos t + C_2 \sin t$ $\quad i = 0 + 1i \quad e^{\alpha t} \cos\beta t = e^0 \cos\beta t$

2) $y'' + 2y' + 8y = 0$

char. eqn. $r^2 + 2r + 8 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{4 - 4\times 8}}{2} = \dfrac{-2 \pm \sqrt{-28}}{2}$

$= -1 \pm \sqrt{7}\, i$ $\quad \sqrt{-28} = \sqrt{28}\cdot\sqrt{-1} = 2\sqrt{7}\cdot i$

Gen. sol'n: $y = C_1 e^{-t} \cos(\sqrt{7}\, t) + C_2 e^{-t} \sin(\sqrt{7}\, t)$

Rmk: No need to worry about the negative branch $1 - \sqrt{7}$, as it will yield precisely the same general solution (use $\cos(-\alpha) = \cos\alpha$, $\sin(-\alpha) = -\sin\alpha$)

3) $y'' - y' + y = 0$

char. eqn.: $r^2 - r + 1 = 0 \Rightarrow r = \dfrac{+1 \pm \sqrt{1 - 4\times 1\times 1}}{2} = \dfrac{1 \pm \sqrt{3}\, i}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$

Gen. sol'n: $y = C_1 e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + C_2 e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$

How the solutions looks like

Recall: $A \cos\theta + B \sin\theta = R \cos(\theta - \varphi)$

$R = \sqrt{A^2 + B^2}$

$\varphi =$ angle of (A, B) on the plane

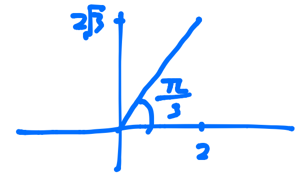
R — Amplitude

φ — Phase

Example: 1) $2\cos\theta + 2\sqrt{3}\sin\theta$

$$= \sqrt{2^2 + (2\sqrt{3})^2}\cos\left(\theta - \frac{\pi}{3}\right)$$

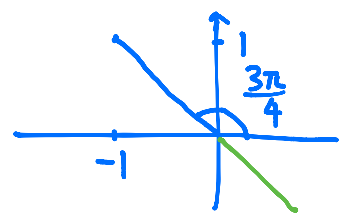
$$= 4\cos\left(\theta - \frac{\pi}{3}\right)$$



2) $-\cos\theta + \sin\theta$.

$$= \sqrt{(-1)^2 + 1^2}\cos\left(\theta - \frac{3\pi}{4}\right)$$

$$= \sqrt{2}\cos\left(\theta - \frac{3\pi}{4}\right)$$



$\arctan\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$

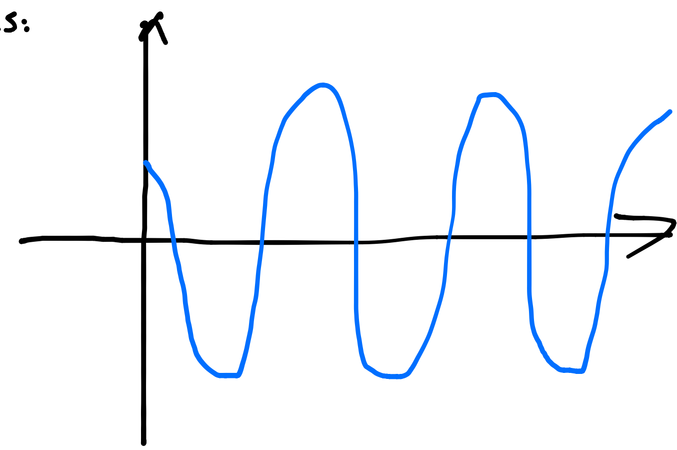
Look at the solutions in previous examples:

1) $y = C_1\cos t + C_2\sin t$

$$= \underbrace{\sqrt{C_1^2 + C_2^2}}\cos(\theta - \varphi)$$

└ Amplitude stays still

$\Rightarrow$ Steady Oscillation

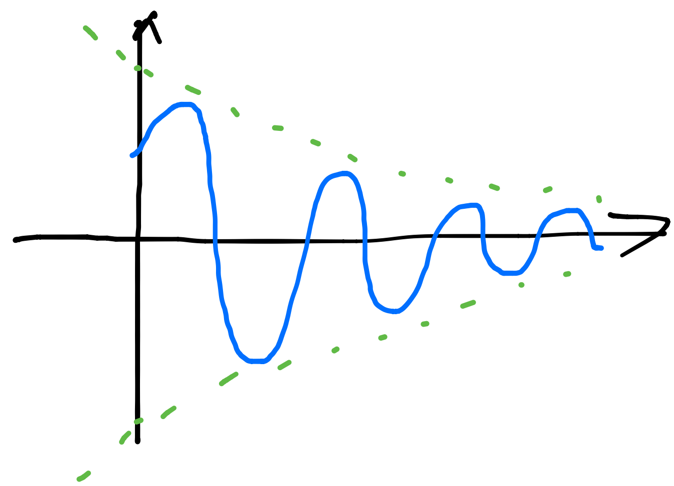


2) $y = C_1 e^{-t}\cos\sqrt{7}t + C_2 e^{-t}\sin\sqrt{7}t$

$$= \underbrace{\sqrt{C_1^2 + C_2^2}\, e^{-t}}\cos(\sqrt{7}t - \varphi)$$

└ Amplitude decays exponentially

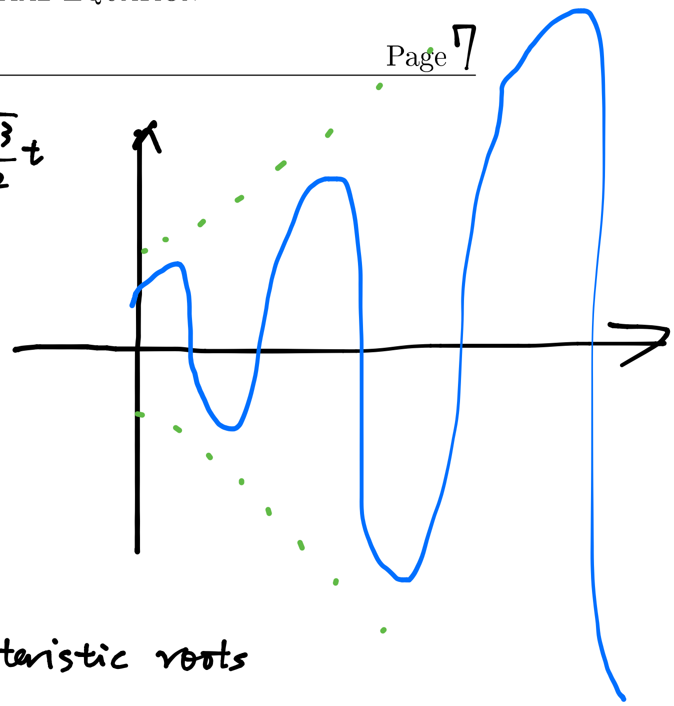
$\Rightarrow$ Decaying Oscillation



3) $y = C_1 e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + C_2 e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

$= \underbrace{\sqrt{C_1^2+C_2^2}\, e^{\frac{1}{2}t}}_{\text{Amplitude grows exponentially}} \cos\left(\frac{\sqrt{3}}{2}t - \varphi\right)$

$\Rightarrow$ Growing Oscillation



In general, if $\alpha \pm i\beta$ are characteristic roots

1) $\alpha = 0 \Rightarrow$ Steady oscillation

2) $\alpha > 0 \Rightarrow$ Growing oscillation

3) $\alpha < 0 \Rightarrow$ Decaying Oscillation.

The above behavior does not depend on the initial values but only on the ODE itself.

Case III: $r_1 = r_2 = r$ i.e., repeated characteristic roots

Formula: $y = C_1 e^{rt} + C_2 t e^{rt}$

(You can obtain this by checking: put te^{rt} into the ODE to see it's a sol'n)

Example: $y'' - 2y' + y = 0$

char. eqn. $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1, 1$

General sol'n: $y = C_1 e^t + C_2 t e^t$

Example: $4y'' - 4y' + y = 0$

char. eqn: $4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0$ Criss-cross

$\Rightarrow r = \frac{1}{2}, \frac{1}{2}$

Gen. sol'n: $y = C_1 e^{\frac{1}{2}t} + C_2 t e^{\frac{1}{2}t}$

Long term behavior:

Example: $y'' + 10y' + 25y = 0, \quad y(0) = 1, \quad y'(0) = \alpha$

char. eqn: $r^2 + 10r + 25 = 0 \Rightarrow (r+5)^2 = 0 \Rightarrow r = -5, -5$

Gen. sol'n: $y = C_1 e^{-5t} + C_2 t e^{-5t}$

$y(0) = 1 \Rightarrow C_1 e^0 + C_2 \cdot 0 \cdot e^0 = 1 \Rightarrow C_1 = 1$

$y'(0) = \alpha \Rightarrow y'(t) = -5C_1 e^{-5t} + C_2\left(1 \cdot e^{-5t} + t(-5)e^{-5t}\right)$

$-5C_1 e^0 + C_2\left(1 \cdot e^0 + 0 \cdot (-5) \cdot e^0\right) = \alpha$

$-5 + C_2 = \alpha \Rightarrow C_2 = \alpha + 5$

Sol'n: $y = e^{-5t} + (\alpha+5) t e^{-5t}$

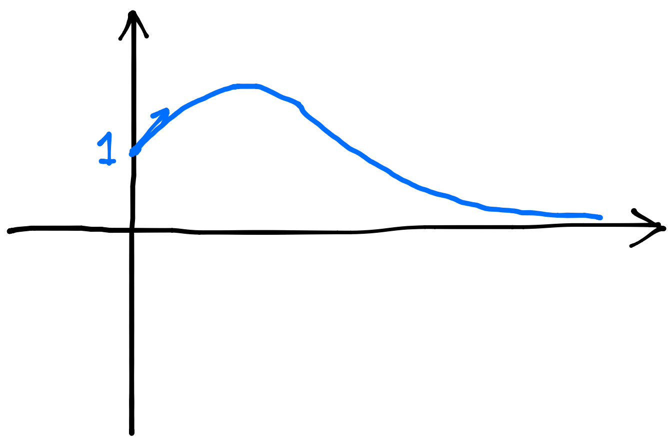
$= \left(1 + (\alpha+5)t\right) e^{-5t}$

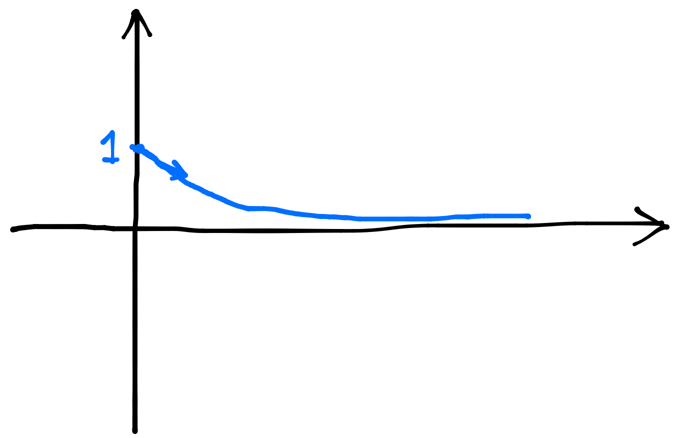
$\lim_{t\to\infty} y(t) = 0$ no matter how α is chosen.

However, there are two ways approaching to zero

If $\alpha + 5 > 0$, $y(t)$ will be eventually positive, i.e., $y(t) \to 0^+$

$\alpha + 5 < 0$, $y(t)$ will be eventually negative, i.e. $y(t) \to 0^-$





$y'(0)=\alpha$

$\alpha > -5$

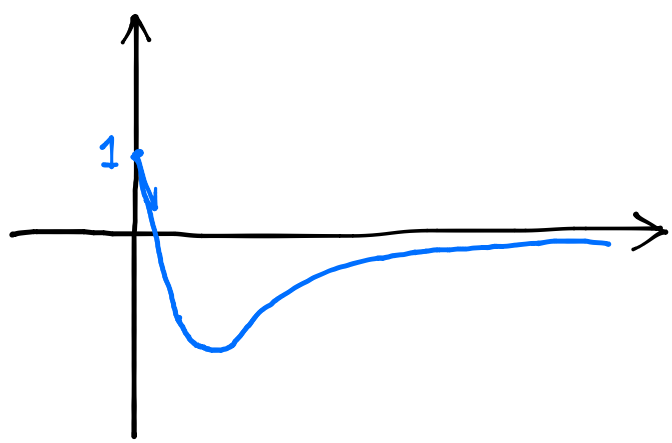
$y(t) = (1+(\alpha+5)t)e^{-5t}$

$\alpha > 0.$

$\alpha > -5$

$y(t) = (1+(\alpha+5)t)e^{-5t}$

$\alpha < 0.$



over shoot

$\alpha < -5$

$y(t) = (1+(\alpha+5)t)e^{-5t}$

Sol'n to the Attendance Quiz

1. $(1)^{\frac{1}{4}} = \left(e^{i2k\pi}\right)^{\frac{1}{4}} = e^{i\frac{k\pi}{2}}$

$$= \begin{cases} e^{i\cdot 0} = 1 \\ e^{i\cdot\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0+1i = i \\ e^{i\cdot 2\cdot\frac{\pi}{2}} = e^{i\pi} = \cos\pi + i\sin\pi = -1+0i = -1 \\ e^{i\cdot 3\cdot\frac{\pi}{2}} = e^{i\frac{3}{2}\pi} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0+(-1)i = -i \end{cases}$$

In short: $(-1)^{\frac{1}{4}} = \pm 1, \pm i$

2. $(2)^{\frac{1}{3}} = \left(2e^{i2k\pi}\right)^{\frac{1}{3}} = \sqrt[3]{2}\cdot e^{i\frac{2k\pi}{3}}$

$$= \begin{cases} \sqrt[3]{2}\, e^{\frac{2\cdot 0\pi}{3}} = \sqrt[3]{2} \\ \sqrt[3]{2}\, e^{\frac{2\pi}{3}} = \sqrt[3]{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \sqrt[3]{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ \sqrt[3]{2}\, e^{\frac{4\pi}{3}} = \sqrt[3]{2}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \sqrt[3]{2}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right) \end{cases}$$

In short: $2^{\frac{1}{3}} = \sqrt[3]{2},\ \sqrt[3]{2}\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)$

Lecture Notes of Differential Equation

Lecture	Page

Lecture Notes of Differential Equation

Lecture	Page